Student Number: ______________________

2015
PRELIMINARY EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 6–9.

Total Marks – 65

Section I Page 2

5 marks

- Attempt Questions 1–5
- Allow about 10 minutes for this section

Section II Pages 3–6

60 marks

- Attempt Questions 6–9
- Allow about 1 hours 50 minutes for this section

Section I

5 marks
Attempt Questions 1–5
Allow about 10 minutes for this section

Use the multiple-choice answer sheet for Questions 1–5.

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1 What is the value of $\sqrt{\dfrac{9.5+3.7}{9.5-3.7}}$, correct to 3 significant figures?

(A) 1.50 (B) 1.508 (C) 1.509 (D) 1.51

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2 Which of the following is equivalent to $x^2 - 5x - 6$?

(A) $(x + 2)(x - 3)$ (B) $(x - 1)(x + 6)$ (C) $(x + 1)(x - 6)$ (D) $(x - 2)(x + 3)$

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3 What are the coordinates of the focus of the parabola $x^2 = 4(y + 1)$?

(A) (0,–1) (B) (0,0) (C) (0,1) (D) (0,4)

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4 Which of the following does **not** represent a function?

(A) $x = 1$ (B) $y = 1$ (C) $x = \sqrt{y}$ (D) $y = \sqrt{x}$

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5 The lines $2x - ky = 0$ and $x - 3y = 0$ are perpendicular. What is the value of k?

(A) $-\frac{1}{6}$ (B) $-\frac{2}{3}$ (C) $-\frac{3}{2}$ (D) –6

Section II

60 marks
Attempt Questions 6–9
Allow about 1 hours 50 minutes for this section

Start each question on a new page. Extra writing pages are available.

All necessary working should be shown in every question.

Question 6 (15 marks) Start a new page

(a) (i) Find the equation of the axis of symmetry of the parabola $y = x^2 - 4x - 5$. **1**

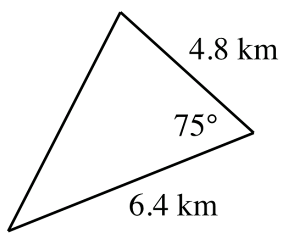
(ii) Find the minimum value of the expression $x^2 - 4x - 5$. **1**

(b) The function $f(x)$ is given by: **2**

$$f(x) = \begin{cases} x^2 - 5 & \text{for } x \leq 2 \\ 7 - 2x & \text{for } x > 2 \end{cases}$$

Find $f(-5) + f(3)$.

(c)



A triangular park has two sides of length 6.4 km and 4.8 km. The angle between these sides is 75°.

(i) Find the length of the third side of the park. Give your answer in kilometres, correct to 1 decimal place. **2**

(ii) Find the area of the park. Give your answer correct to the nearest square kilometre. **1**

(d) Solve the following inequations.

(i) $2x^2 - x - 28 < 0$ **2**

(ii) $|2x - 1| > 3$ **2**

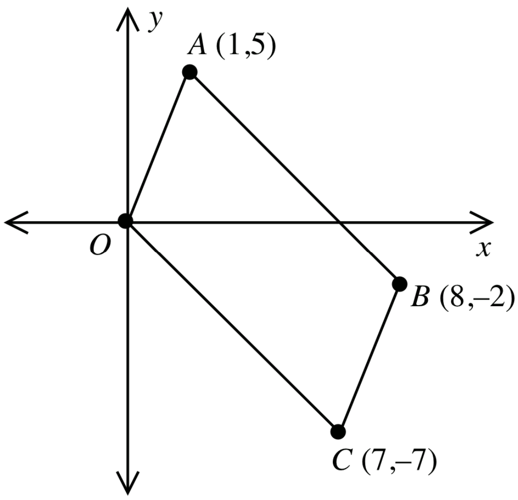
Question 6 (continued)

(e) (i) Draw a large, neat sketch of the function $y = \sqrt{4 - x^2}$. Indicate where the curve crosses the coordinate axes. **2**

(ii) Write down the domain and range of the function $y = \sqrt{4 - x^2}$. **2**

Question 7 (15 marks) Start a new page

(a) In the diagram below, $O(0,0)$, $A(1,5)$, $B(8,–2)$ and $C(7,–7)$ are the vertices of the quadrilateral $OABC$.



(i) Find the midpoint of the interval joining AC. **1**

(ii) Find the gradient of AB. **1**

(iii) Show that the equation of AB is $x + y - 6 = 0$. **1**

(iv) Find the exact length of AB. **2**

(v) Show that AB is parallel to OC. **1**

(vi) What type of quadrilateral is $OABC$? Use appropriate reasoning and calculations to justify your answer. **2**

(vii) Find the perpendicular distance from O to AB. Give your answer in exact form. **2**

(viii) Hence find the area of quadrilateral $OABC$. Give your answer in exact form. **2**

(b) Find the values of A, B and C, given that $A(x^2 + 1)^2 + B(x^2 + 1) + C \equiv 2x^4 + 3x^2 - 3$. **3**

Question 8 (15 marks) Start a new page

(a) $A(-2,5)$ and $B(1,1)$ are two points. Find the coordinates of the point $P(x,y)$ such that B is the midpoint of AP. **2**

(b) (i) Show that $(p + q)^2 - 4pq = (p - q)^2$. **1**

(ii) Prove that the roots of the equation $px^2 - (p + q)x + q = 0$ are rational. **2**

The quadratic equation $2x^2 + 4x - 1 = 0$ has roots α and β. Find the values of:

(iii) $\alpha + \beta$ **1**

(iv) $\alpha\beta$ **1**

(v) $|\alpha - \beta|$ **2**

(c) Consider the function $f(x) = 2^x + 2^{-x}$.

(i) Show algebraically that $f(x)$ is an even function. **2**

(ii) Sketch the graph of $y = f(x)$. **1**

(d) Show that the line $3x + 4y - 9 = 0$ is a tangent to the circle $(x - 1)^2 + (y + 1)^2 = 4$. **3**

Question 9 (15 marks) Start a new page

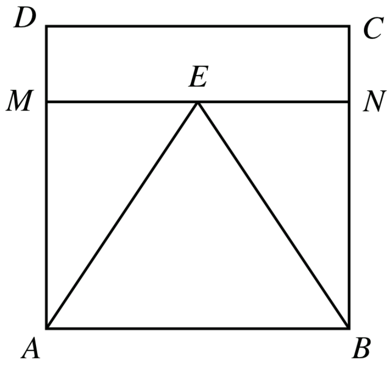
(a) Solve the equation $\sqrt{3}\tan\theta + 1 = 0$ for $0° \le \theta \le 360°$. **2**

(b) Find the value of m in the quadratic equation $2x^2 - 27x + m = 0$ if one root is double the other. **2**

(c) Prove that $\sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta = \sin^2\alpha - \sin^2\beta$. **3**

Question 9 (continued)

(d)



In the diagram above, $ABCD$ is a square of side 2 cm. E is the point inside the square such that ΔABE is an equilateral triangle. The line MN through E is perpendicular to both AD and BC.

(i) Copy the diagram onto your answer sheet.

(ii) Show that $\angle ECN = 75°$. **3**

(iii) Show that $\tan 75° = 2 + \sqrt{3}$. **2**

(e) Solve the equation $(x^2 - x - 10)(x^2 - x - 16) = 40$. **3**

End of paper

2 UNIT MATHEMATICS
2015 PRELIMINARY EXAMINATION

SECTION I

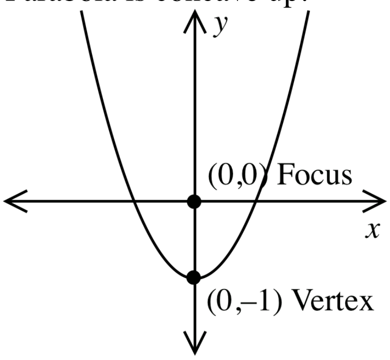
1 $\sqrt{\dfrac{9.5+3.7}{9.5-3.7}} = \sqrt{\dfrac{13.2}{5.8}}$ 1. **D**

$= 1.50859606$
≈ 1.51

2 $x^2 - 5x - 6 = (x+1)(x-6)$ 2. **C**

3 $x^2 = 4(y+1)$ 3. **B**

$x^2 = 4.1.(y+1)$
$\therefore$ Focal length = 1 unit
Parabola is concave up.



Vertex $\equiv (0,-1)$
$\therefore$ Focus $\equiv (0,0)$

4 The graph of $x = 1$ represents a vertical line. 4. **A**
For this value of x there is more than one corresponding value of y.
$\therefore$ $x = 1$ is not a function.

5 5. **B**

Gradient of line 1 $= \dfrac{-a}{b} = \dfrac{-2}{-k} = \dfrac{2}{k}$

Gradient of line 2 $= \dfrac{-a}{b} = \dfrac{-1}{-3} = \dfrac{1}{3}$

Since the lines are perpendicular we have:

$$m_1 \times m_2 = -1$$
$$\frac{2}{k} \times \frac{1}{3} = -1$$
$$\frac{2}{3k} = -1$$
$$2 = -3k$$
$$k = -\tfrac{2}{3}$$

SECTION II

QUESTION 6

(a) (i) Equation of axis of symmetry is:

$$x = \frac{-b}{2a}$$
$$= \frac{-(-4)}{2(1)}$$
$$= 2$$

(ii) When $x = 2$:

$$y = (2)^2 - 4(2) - 5$$
$$= 4 - 8 - 5$$
$$= -9$$

∴ Minimum value = –9

(b) $f(-5) + f(3) = [(-5)^2 - 5] + [7 - 2(3)]$
$= [25 - 5] + [7 - 6]$
$= 20 + 1$
$= 21$

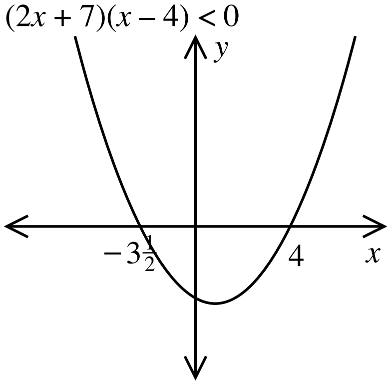
(c) (i) Let d be the length of the third side.

$$d^2 = 4.8^2 + 6.4^2 - 2 \times 4.8 \times 6.4 \times \cos 75°$$
$$= 48.09815737$$
$$d = 6.935283546$$

∴ The length of the third side is 6.9 km.

(ii) Area $= \frac{1}{2} \times 4.8 \times 6.4 \times \sin 75°$
$= 14.83662069$
$= 15 \text{ km}^2$

(d) (i) $2x^2 - x - 28 < 0$
$(2x + 7)(x - 4) < 0$



$\therefore -3\frac{1}{2} < x < 4$

(ii) When $2x - 1 \geq 0$ (or $x \geq \frac{1}{2}$):

$$2x - 1 > 3$$
$$2x > 4$$
$$x > 2$$

Since $x \geq \frac{1}{2}$, the solution $x > 2$ is valid.

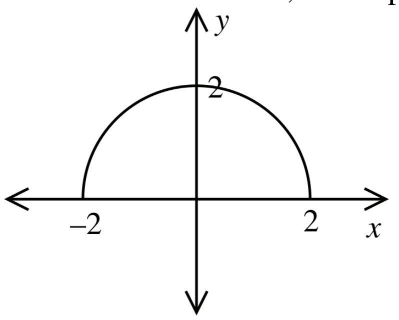
When $2x - 1 < 0$ (or $x < \frac{1}{2}$):

$$-(2x - 1) > 3$$
$$-2x + 1 > 3$$
$$-2x > 2$$
$$x < -1$$

Since $x < \frac{1}{2}$, the solution $x < -1$ is valid.

$\therefore$ The solution is $x < -1$ and $x > 2$.

(e) (i) Curve is a semi-circle, on the positive side of the x-axis.



(ii) Domain: $\{x: -2 \leq x \leq 2\}$
Range: $\{y: 0 \leq y \leq 2\}$

QUESTION 7

(a) (i) Let M be the midpoint of the interval AC.

$$M \equiv \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$\equiv \left(\frac{1 + 7}{2}, \frac{5 - 7}{2}\right)$$
$$\equiv (4, -1)$$

(ii) Gradient of $AB = \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{-2 - 5}{8 - 1}$$
$$= \frac{-7}{7}$$
$$= -1$$

(iii) Equation of AB:

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$
$$\frac{y-5}{x-1}=\frac{-2-5}{8-1}$$
$$\frac{y-5}{x-1}=\frac{-7}{7}$$
$$\frac{y-5}{x-1}=-1$$
$$y-5=-1(x-1)$$
$$y-5=-x+1$$
$$x+y-6=0$$

(iv)

$$AB^2=(x_1-x_2)^2+(y_1-y_2)^2$$
$$=(1-8)^2+(5+2)^2$$
$$=7^2+7^2$$
$$=98$$
$$AB=\sqrt{98}$$
$$=7\sqrt{2}\text{ units}$$

(v) Gradient of $AB = -1$

Gradient of $OC = \dfrac{y_2-y_1}{x_2-x_1}$

$$=\frac{0+7}{0-7}$$
$$=\frac{7}{-7}$$
$$=-1$$

$\therefore$ AB is parallel to OC

(vi) Let N be the midpoint of the interval OB.

$$N\equiv\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$
$$\equiv\left(\frac{0+8}{2},\frac{0-2}{2}\right)$$
$$\equiv(4,-1)$$

Since M and N are the same point, the diagonals bisect each other.

$\therefore$ $OABC$ is a parallelogram.

(vii)

$$\text{Distance}=\left|\frac{ah+bk+c}{\sqrt{a^2+b^2}}\right|$$
$$=\left|\frac{1.0+1.0-6}{\sqrt{1^2+1^2}}\right|$$
$$=\left|\frac{-6}{\sqrt{2}}\right|$$
$$=\frac{6}{\sqrt{2}}\text{ units}$$

(viii) Since $OABC$ is a parallelogram,

$$\begin{aligned} \text{Area} &= bh \\ &= 7\sqrt{2} \times \frac{6}{\sqrt{2}} \\ &= 42 \text{ square units} \end{aligned}$$

(b) $$\begin{aligned} 2x^4 + 3x^2 - 3 &\equiv A(x^2 + 1)^2 + B(x^2 + 1) + C \\ &\equiv A(x^4 + 2x^2 + 1) + B(x^2 + 1) + C \\ &\equiv Ax^4 + 2Ax^2 + A + Bx^2 + B + C \\ &\equiv Ax^4 + (2A + B)x^2 + (A + B + C) \end{aligned}$$

$A = 2$

$$\begin{aligned} 2A + B &= 3 \\ 2(2) + B &= 3 \\ 4 + B &= 3 \\ B &= -1 \end{aligned}$$

$$\begin{aligned} A + B + C &= -3 \\ 2 - 1 + C &= -3 \\ 1 + C &= -3 \\ C &= -4 \end{aligned}$$

$\therefore A = 2, B = -1$ and $C = -4$

QUESTION 8

(a) We have:

$$\begin{aligned} \frac{-2 + x}{2} &= 1 \\ -2 + x &= 2 \\ x &= 4 \end{aligned}$$

and:

$$\begin{aligned} \frac{5 + y}{2} &= 1 \\ 5 + y &= 2 \\ y &= -3 \end{aligned}$$

$\therefore P \equiv (4,-3)$

(b) (i) $$\begin{aligned} (p + q)^2 - 4pq &= p^2 + 2pq + q^2 - 4pq \\ &= p^2 - 2pq + q^2 \\ &= (p - q)^2 \end{aligned}$$

(ii) For rational roots, we need the discriminant to be a perfect square.

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= [-(p + q)]^2 - 4 \times p \times q \\ &= (p + q)^2 - 4pq \\ &= (p - q)^2 \end{aligned}$$

$\therefore$ The roots of the equation are rational.

(iii)

$$\alpha+\beta=\frac{-b}{a}$$
$$=\frac{-4}{2}$$
$$=-2$$

(iv)

$$\alpha\beta=\frac{c}{a}$$
$$=\frac{-1}{2}$$
$$=-\tfrac{1}{2}$$

(v)

$$(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$$
$$=(-2)^2-4(-\tfrac{1}{2})$$
$$=4+2$$
$$=6$$

$$|\alpha-\beta|=\left|\sqrt{(\alpha-\beta)^2}\right|$$
$$=\left|\sqrt{6}\right|$$
$$=\sqrt{6}$$

(c) (i) $f(x)=2^x+2^{-x}$

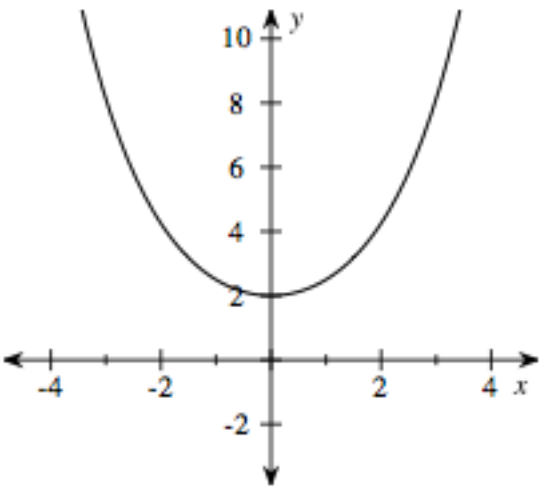
$f(-x)=2^{(-x)}+2^{-(-x)}$

$=2^{-x}+2^{x}$

$=f(x)$

$\therefore f(x)$ is an even function

(ii)



(d) We have:

$(x-1)^2 + (y+1)^2 = 4$ ... [1]

and:

$$3x+4y-9=0$$
$$3x=9-4y$$
$$x=3-\tfrac{4}{3}y \ldots [2]$$

Substituting [2] into [1] gives:

$$(3-\tfrac{4}{3}y-1)^2+(y+1)^2=4$$
$$(2-\tfrac{4}{3}y)^2+(y+1)^2=4$$
$$4-\tfrac{16}{3}y+\tfrac{16}{9}y^2+y^2+2y+1=4$$
$$\tfrac{25}{9}y^2-\tfrac{10}{3}y+5=4$$
$$\tfrac{25}{9}y^2-\tfrac{10}{3}y+1=0$$
$$25y^2-30y+9=0$$
$$(5y+3)^2=0$$

Since there is only one solution to this equation, the straight line only touches the circle in one point.

∴ The line is a tangent to the circle.

or

Centre of circle is (1,–1) and radius is 2 units.

Using perpendicular distance formula:

$$d=\left|\frac{ah+bk+c}{\sqrt{a^2+b^2}}\right|$$
$$=\left|\frac{3.1+4.-1-9}{\sqrt{3^2+4^2}}\right|$$
$$=\left|\frac{3-4-9}{\sqrt{25}}\right|$$
$$=\left|\frac{-10}{5}\right|$$
$$=2$$

Since this is equal to the radius of the circle, the line $3x+4y-9=0$ only touches the circle in one point.

∴ The line is a tangent to the circle.

QUESTION 9

(a)

$$\sqrt{3}\tan\theta+1=0$$
$$\sqrt{3}\tan\theta=-1$$
$$\tan\theta=-\tfrac{1}{\sqrt{3}}$$
$$\theta=(180-30)^\circ \text{ or } (360-30)^\circ$$
$$\theta=150^\circ \text{ or } 330^\circ$$

(b) Let the roots be α and 2α.
Therefore:

$$\alpha + 2\alpha = \frac{-b}{a}$$
$$3\alpha = \frac{-(-27)}{2}$$
$$= \tfrac{27}{2}$$
$$\alpha = \tfrac{9}{2}$$

and:

$$\alpha \times 2\alpha = \frac{c}{a}$$
$$2\alpha^2 = \frac{m}{2}$$
$$m = 4\alpha^2$$
$$= 4(\tfrac{9}{2})^2$$
$$= 81$$

(c) LHS = $\sin^2\alpha \cos^2\beta - \cos^2\alpha \sin^2\beta$
$= \sin^2\alpha\,(1 - \sin^2\beta) - (1 - \sin^2\alpha)\sin^2\beta$
$= \sin^2\alpha\,(1 - \sin^2\beta) - \sin^2\beta\,(1 - \sin^2\alpha)$
$= \sin^2\alpha - \sin^2\alpha \sin^2\beta - \sin^2\beta + \sin^2\alpha \sin^2\beta$
$= \sin^2\alpha - \sin^2\beta$
= RHS

(d) (ii) $AB = BC = 2$ cm (Equal sides of square $ABCD$)
$AB = EB = 2$ cm (Equal sides of equilateral ΔABE)
$\therefore\ BC = EB = 2$ cm
$\therefore\ \Delta EBC$ is isosceles
$\therefore\ \angle CEB = \angle ECB$ (Equal angles opposite equal sides)
Since $\angle EBC = 30°$, we have: (Angle sum of triangle)

$$\angle ECB + \angle CEB + 30° = 180°$$
$$2\angle ECB + 30° = 180°$$
$$2\angle ECB = 150°$$
$$\angle ECB = 75°$$

(iii) Since E is the midpoint of MN we have $ME = EN = 1$ cm.
Using Pythagoras' theorem:

$$EB^2 = EN^2 + NB^2$$
$$2^2 = 1^2 + NB^2$$
$$4 = 1 + NB^2$$
$$NB^2 = 3$$
$$NB = \sqrt{3}$$

Therefore:

$$CN = CB - NB$$
$$= 2 - \sqrt{3}$$

Now:

$$\tan\angle ECN = \frac{EN}{CN}$$

$$\tan 75° = \frac{1}{2-\sqrt{3}}$$

$$= \frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$= \frac{2+\sqrt{3}}{4-3}$$

$$= 2+\sqrt{3}$$

(e) $(x^2 - x - 10)(x^2 - x - 16) = 40$

Let $t = x^2 - x$

$$(t-10)(t-16) = 40$$

$$t(t-16) - 10(t-16) = 40$$

$$t^2 - 16t - 10t + 160 = 40$$

$$t^2 - 26t + 120 = 0$$

$$(t-20)(t-6) = 0$$

$$t = 6 \text{ or } 20$$

Therefore:

$$x^2 - x = 6$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

$$x = 3 \text{ or } -2$$

and:

$$x^2 - x = 20$$

$$x^2 - x - 20 = 0$$

$$(x-5)(x+4) = 0$$

$$x = 5 \text{ or } -4$$

$\therefore$ Solutions are $x = -4, -2, 3$ and 5.